STAT0041: Stochastic Calculus

Lecture 4 - Optional Stopping Theorem

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Key concepts:

- *Doob's optional stopping theorem;*
- *Doob's martingale convergence theorem.*

4.1 Doob's optional stopping theorem

Considering gambling model in example 3.1. Gamblers always adopt an exit strategy based on prior history such as withdraw when one win $500 in order to make a profit, or when one lose 20% of initial capital. This exit time is a stopping time. We go back to martingale betting strategy example 3.5

Example 4.1 (Martingale betting strategy (2)) *Define* $\tau := \inf\{n \geq 1, \eta_n = 1\}$*, then*

$$\mathrm{P}(\tau = n) = \frac{1}{2^n}, \mathrm{P}(\tau < \infty) = 1,$$

$$\mathbb{E}\tau = \sum_{k=1}^{\infty} k\mathrm{P}(\tau = k) = \sum_{k=1}^{\infty} k\frac{1}{2^k} = 2 < \infty.$$

And

$$\xi_\tau = 1,\ \mathbb{E}\xi_\tau \neq \mathbb{E}\xi_0 = 0.$$

where $\xi_\tau(\omega) := \xi_{\tau(\omega)}(\omega)$*. This contradicts "fair" gambling. Notice that*

$$\mathbb{E}(|\xi_n|\cdot\mathbf{1}_{\{\tau>n\}}) = \mathbb{E}((2^n - 1)\mathbf{1}_{\{\tau>n\}}) = 1 - \frac{1}{2^n}.$$

It shows although the probability that this gambling lasts for a long time is small, the amount of debt $|\xi_n| = 2^n - 1$ *after a long time is very large. The gambler's debt before his first profit is*

$$\xi_{\tau-1} = \begin{cases} \xi_0 = 0, & \tau = 1 \\ \xi_0 - f_1(\xi_0) = -1, & \tau = 2 \\ \xi_0 - f_1(\xi_0) - f_2(\xi_0, -1) = -3, & \tau = 3 \\ \dots & \dots \\ \xi_0 - f_1(\xi_0) - \cdots - f_{n-1}(\xi_0, \underbrace{-1, \cdots, -1}_{n-2}) = -(2^{n-1} - 1), & \tau = n \end{cases}.$$

Then

$$\begin{aligned}
\mathbb{E}\xi_{\tau-1} &= \sum_{k=1}^{\infty} \mathbb{E}\xi_{k-1}\mathbf{1}_{\{\tau=k\}} \\
&= \sum_{k=1}^{\infty} -(2^{k-1}-1)\mathrm{P}(\tau=k) \\
&= \sum_{k=1}^{\infty} -(2^{k-1}-1)\frac{1}{2^k} \\
&= -\infty.
\end{aligned}$$

This suggests that, on average, the gambler's debt is infinite before the first profit. It can be intuitively understood that the gambler needs to have infinite wealth at the initial time in order to ensure the "fairness" of the gambling with martingale betting strategy.

Theorem 4.2 (Doob's sampling theorem) *Let (X_n) be a discrete-time martingale and τ be a stopping time with values in $\mathbb{N} \cup \{+\infty\}$, both with respect to a filtration $(\mathscr{F}_n)$, then the stopped process X^τ is a martingale.*

Moreover, if stopping time τ is bounded, then $\mathbb{E}[X_\tau] = \mathbb{E}[X_0]$.

Doob's sampling theorem 4.2 shows that it is not possible to increase the expectation of fortune using a bounded stopping time strategy in a "fair" game.

Theorem 4.3 (Doob's optional stopping theorem) *Let (X_n) be a discrete-time martingale and $\sigma \le \tau < T < \infty$ be two almost surely bounded stopping times with values in $\mathbb{N} \cup \{+\infty\}$, both with respect to a filtration $(\mathscr{F}_n)$, then*

$$\mathbb{E}[X_\tau|\mathscr{F}_\sigma] = X_\sigma.$$

Remark 4.4 (Optional stopping theorem for uniformly integrable martingales) *Let (X_n) be a uniformly integrable martingale, and let X_∞ be the a.s. limit of (X_n) when $n \to \infty$. Then, for every choice of the stopping times τ, σ such that $\tau \le \sigma$, we have $X_\sigma \in L^1$ and*

$$X_\tau = E[X_\sigma \mid \mathscr{F}_\tau]$$

uniformly integrable *refers to appendix C of [1] and section 3.4 of [2]*

Remark 4.5 *Conversely, if a game is impossible to increase the expectation of fortune for any bounded stopping time strategy, then intuitively the game should be "fair", i.e., a martingale. We have following conclusion:*

Let (X_n) be a adapted process and $\mathbb{E}|X_n| < \infty, \forall n$, then (X_n) is a martingale if and only if for all almost surely bounded stopping times $\tau \le \sigma$

$$\mathbb{E}[X_\tau] = \mathbb{E}[X_\sigma].$$

Proposition 4.6 (Doob's maximal martingale inequality) *Let (X_n) be a submartingale, then for all $c > 0$ and $N \in \mathbb{N}$,*

$$c \cdot \mathrm{P}(\max_{0 \le n \le N} X_n \ge c) \le \mathbb{E}[X_N^+].$$

Remark 4.7 *Let (X_n) be a martingale, and for some $p \ge 1$, $\mathbb{E}|X_n|^p < \infty$, $\forall n$. Then for all for all $c > 0$ and $N \in \mathbb{N}$,*

$$\mathrm{P}(\max_{0 \le n \le N} |X_n| \ge c) \le \frac{\mathbb{E}|X_N|^p}{c^p}.$$

4.2 Doob's martingale convergence theorem

Let (ξ_n) be a $(\mathscr{F}_n)$ adapted process, $a < b$, define a sequence of stopping time:

$$\begin{aligned}
\tau_0 &= 0 \\
\tau_1 &= \inf\{n \ge 0, \xi \le a\}, \\
\tau_2 &= \inf\{n \ge \tau_1, \xi \ge b\}, \\
\tau_3 &= \inf\{n \ge \tau_2, \xi \le a\}, \\
\tau_4 &= \inf\{n \ge \tau_3, \xi \ge b\}, \\
&\cdots \\
\tau_{2m-1} &= \inf\{n \ge \tau_{2m-2}, \xi_n \le a\}, \\
\tau_{2m} &= \inf\{n \ge \tau_{2m-1}, \xi_n \ge b\}.
\end{aligned}$$

Define the number of ξ_n upcrossing before the moment N:

$$U_N[a,b] = \begin{cases} 0, & \tau_2 > N, \\ \max\{m, \tau_{2m} \le N\}, & \tau_2 \le N. \end{cases}$$

Lemma 4.8 (Doob's upcrossing inequality) *Let $(\xi_n, \mathscr{F}_n)$ be a submartingale, then for all $N \ge 1$*

$$\mathbb{E}U_N[a,b] \le \frac{\mathbb{E}(\xi_N - a)^+}{b-a}.$$

Theorem 4.9 (Doob's martingale convergence theorem) *Let $(\xi_n, \mathscr{F}_n)$ be a submartingale satisfying $\sup_n \mathbb{E}|\xi_n| < \infty$, then*

$$\xi_\infty := \lim_{n \to \infty} \xi_n, \ a.s.$$

exists, and $\mathbb{E}|\xi_\infty| < \infty$.

References

[1] Oksendal, Bernt. Stochastic differential equations: an introduction with applications. Springer Science & Business Media, 2013.

[2] Le Gall, Jean-François. Brownian motion, martingales, and stochastic calculus. Springer International Publishing Switzerland, 2016.